Tutorial – The Constraint Satisfaction Problem Dichotomy Theorem. Lecture 1

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Waterloo (Canada)

Assoc. Sym. Logic meeting – Ames, IA
15 May 2024

Plan

1. (Today) The CSP dichotomy theorem (Bulatov & Zhuk).
    - Constraint satisfaction problems
    - Statement of the Dichotomy Theorem
    - “Algebraic” perspective
    - Hopefully accessible to everyone.
2. (Tomorrow) Algebraic idea # 1 from Zhuk’s proof
    - Still relatively accessible, but more technical. (Bring coffee)
3. (Friday) Algebraic idea # 2 from Zhuk’s proof
    - Very technical, assumes some universal algebra. (You’ve been warned)

Part 1 – Constraint Satisfaction Problems

M fixed structure: relational, finite, and finite signature.

φ formula over **M**

- $\wedge$at-fmla – conjunction of atomic formulas
- pp-fmla – $\exists\vec{y}\psi$ where ψ is $\wedge$at

$\varphi^{\mathbf{M}}$ – the *n*-ary relation defined in **M** by $\varphi(x_1, \ldots, x_n)$.

Fine print: formulas may contain parameters from **M**.

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Constraint Satisfaction Problem $\mathrm{CSP}_p(\mathbf{M})$

(Fix **M**.) $\mathrm{CSP}_p(\mathbf{M})$ is the following decision problem:

Input: $\wedge$at-fmla φ (in signature of **M**)

Question: Is $\varphi^{\mathbf{M}} \neq \varnothing$?

Fine print: formulas may contain parameters from **M**.

CSP$_p$($\mathbf{M}$) can be easy or hard

Example 1: $\mathbf{M}_{3SAT} = (\{0,1\}, R_{3SAT})$ where

$$R_{3SAT} = \{(x_1,\ldots,x_6) : (x_1,x_2,x_3) \neq (x_4,x_5,x_6)\}.$$

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$$\begin{aligned} &R_{3SAT}(x,y,z,0,0,0) \quad \text{encodes} \quad x \vee y \vee z \\ &R_{3SAT}(x,y,z,0,0,1) \quad \text{encodes} \quad x \vee y \vee \neg z, \quad \textit{etc.} \end{aligned}$$

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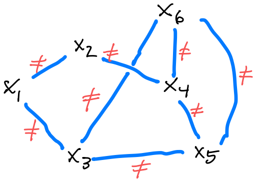
Instances of 3-SAT can be encoded as $\wedge$at-fmlas over $\mathbf{M}_{3SAT}$.

$\therefore$ we have a poly-time reducton 3-SAT $\leq_P$ $\mathrm{CSP}_p(\mathbf{M}_{3SAT})$.

$\therefore$ $\mathrm{CSP}_p(\mathbf{M}_{3SAT})$ is NP-hard, hence NP-complete.

Example 2: $\mathbf{K}_3 = (\{0,1,2\}, \neq)$.

(=-free, parameter-free) $\bigwedge$at-fmlas in this signature can be pictured; e.g.,



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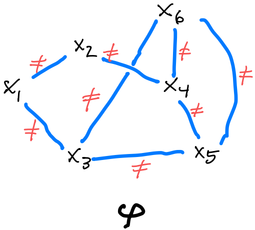

$$\varphi^{\mathbf{K}_3} \neq \varnothing \iff \exists \text{ assignment } \{x_1, \ldots, x_6\} \to \{0,1,2\} \text{ preserving } \neq$$
$$\iff \text{this graph can be 3-colored.}$$

$\rightsquigarrow$ polytime reduction 3-COL $\leq_P$ CSP$_p(\mathbf{K}_3)$.

$\therefore$ CSP$_p(\mathbf{K}_3)$ is NP-complete.

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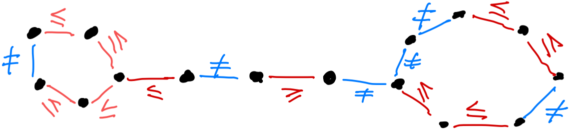

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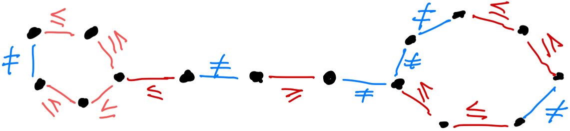


We can efficiently test whether any such configurations occur in φ.

$\therefore$ $\mathrm{CSP}_p(\mathbf{K}_{2,\leq})$ is in P.

Example 4: $\mathbf{M}_{3lin} = (\{0,1,2\},\ R)$ where

$$R = \{(x,y,z,w) : x - y + z = w \pmod 3\}.$$

Atomic formulas over $\mathbf{M}_{3lin}$ express (short) linear equations/$\mathbb{Z}_3$:

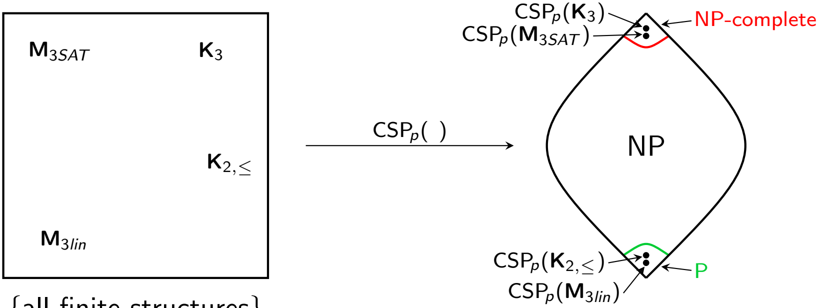
$$\begin{array}{rr} R(x,y,z,w) & x - y + z - w = 0 \\ R(x,y,z,1) & x - y + z = 1,\ \text{etc} \end{array}$$

So $\wedge$at-fmlas over $\mathbf{M}_{3lin}$ express (certain) systems of linear equations/$\mathbb{Z}_3$.

We can solve such systems in poly time.

$\therefore$ $\mathrm{CSP}_p(\mathbf{M}_{3lin})$ is in P.

Part 2 – The Dichotomy Theorem



$\mathbf{M}_{3SAT}$

$\mathbf{K}_3$

$\mathbf{K}_{2,\leq}$

$\mathbf{M}_{3lin}$

$\{$all finite structures$\}$

$\mathrm{CSP}_p(\ )$

$\mathrm{CSP}_p(\mathbf{K}_3)$

$\mathrm{CSP}_p(\mathbf{M}_{3SAT})$

NP-complete

NP

$\mathrm{CSP}_p(\mathbf{K}_{2,\leq})$

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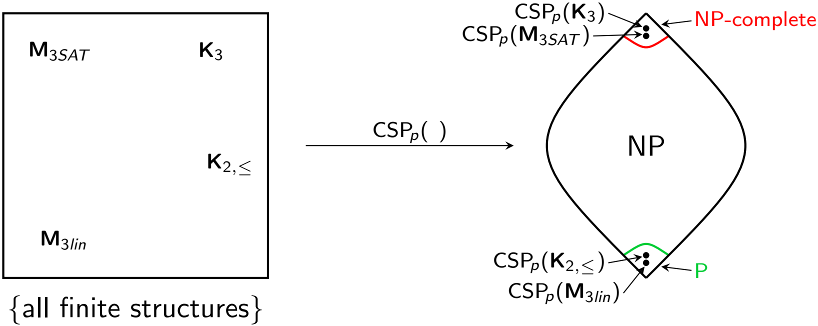
P



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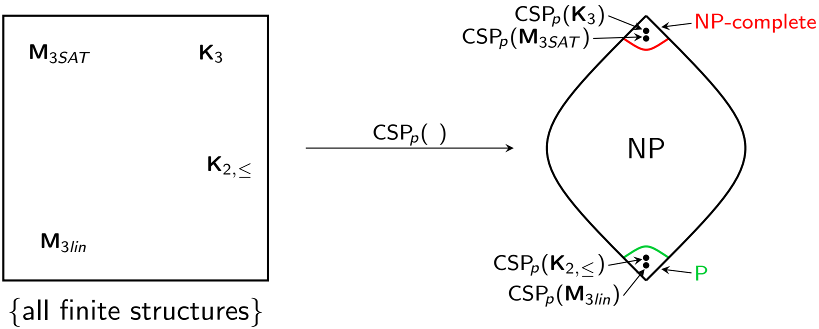
CSP Dichotomy Conjecture (Feder, Vardi 1998)

For every $\mathbf{M}$, $\text{CSP}_p(\mathbf{M})$ is in P or is NP-complete.



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CSP Dichotomy Conjecture (Feder, Vardi 1998)

For every $\mathbf{M}$, $\mathrm{CSP}_p(\mathbf{M})$ is in P or is NP-complete.

Plausible (in 1998).

- Known for 2-element structures (Schaefer 1978)
- Known for core graphs (Hell, Nešetřil 1990)

(Where should the "dividing line" be?)

pp-interpretations

There is one "obvious" reason for $\mathrm{CSP}_p(\mathbf{M})$ to be NP-complete:

$$\text{If } \mathbf{M}_{3SAT} \text{ (or } \mathbf{K}_3\text{) is } \boxed{\text{pp-interpretable}} \text{ in } \mathbf{M}.$$

"pp-interpretation" means the usual thing:

> There is a pp-definable set $D \subseteq \mathbf{M}^n$, a pp-definable equivalence relation E on D with two blocks (so $E \subseteq \mathbf{M}^{2n}$), and a pp-definable 6-ary relation R on D (so $R \subseteq \mathbf{M}^{6n}$) such that
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> $$(D/E,\ R/E) \cong \mathbf{M}_{3SAT}.$$

(A.k.a. "gadget definition.")

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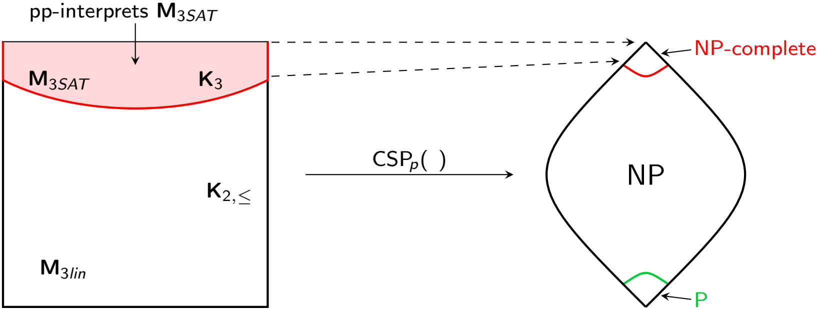
Easy Fact

If $\mathbf{M}_{3SAT} \overset{pp}{\hookrightarrow} \mathbf{M}$, then $\mathrm{CSP}_p(\mathbf{M})$ is NP-complete.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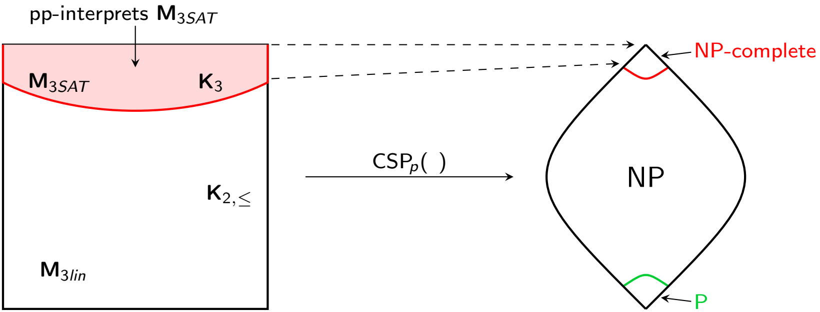

Refined Dichotomy Conjecture (Bulatov, Jeavons, Krokhin 2001)

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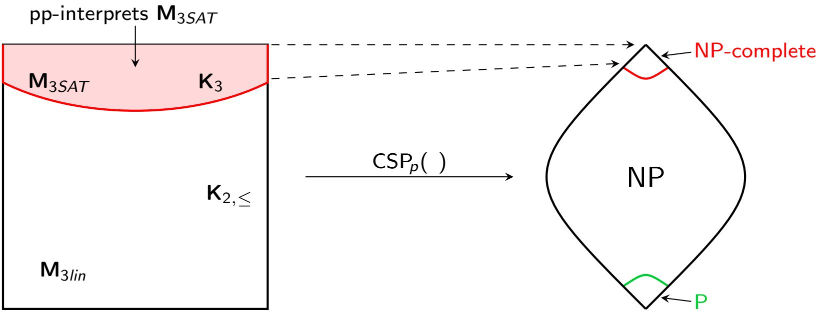

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If $\mathbf{M}_{3SAT} \overset{pp}{\not\hookrightarrow} \mathbf{M}$, then $\mathrm{CSP}_p(\mathbf{M})$ is in P.

The race is on!

Lots of partial results!

Frenetic activity!

Conferences! Workshops! Grant money! And then . . .

Part 3 – The Dichotomy Theorem

The Refined Conjecture is proved!

CSP Dichotomy Theorem (A. Bulatov, D. Zhuk 2017; 2020.)

If $\mathbf{M}$ is finite and $\mathbf{M}_{3SAT} \overset{pp}{\not\to} \mathbf{M}$, then $\mathrm{CSP}_p(\mathbf{M})$ is in P.

It was fun while it lasted.

Part 4 – The algebraic perspective

Example: $\mathbf{M} = (M, R)$ with arity$(R) = 2$.

Endomorphism of $\mathbf{M}$: any map $f : M \to M$ satisfying

$$\binom{a}{b} \in R \implies \binom{f(a)}{f(b)} \in R.$$

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Definition

A **polymorphism** of $\mathbf{M}$ is any map $f : M^n \to M$ satisfying

$$\begin{pmatrix} a_1 \\ b_1 \end{pmatrix}, \dots, \begin{pmatrix} a_n \\ b_n \end{pmatrix} \in R \implies \begin{pmatrix} f(a_1, \dots, a_n) \\ f(b_1, \dots, b_n) \end{pmatrix} \in R.$$

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Example: monotone boolean functions = polymorphisms of $(\{0, 1\}, \leq)$.

(Similarly for relations of higher arity, or **M** with more than one relation.)

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- Any "interesting" 3-ary polymorphisms? Yes!!

$$\text{majority}(x,y,z).$$

On the other hand, $\mathbf{M}_{3SAT} = (\{0,1\}, R_{3SAT})$ where

$$R_{3SAT} = \{(x_1, \dots, x_6) : (x_1, x_2, x_3) \neq (x_4, x_5, x_6)\}$$

has only “trivial” polymorphisms (of all arities):

projections composed with an automorphism.

The same is true of $\mathbf{K}_3 = (\{0,1,2\}, \neq)$.

The algebra of a finite structure

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M for the structure; $\mathbb{M}$ for its associated algebra.

Example:

$$\mathbf{M} = (\{0,1\}, \leq)$$
$$\mathbb{M} = (\{0,1\}, \{\text{all nonconstant monotone boolean functions}\}).$$

Fix **M**. $\mathbb{M}$ its idempotent polymorphism algebra.

Each basic relation (say k-ary) of **M**:

- is preserved (coordinate-wise) by all operations of $\mathbb{M}$ ...
- ...so is a **subuniverse** of $\mathbb{M}^k$.

Same is true for pp-definable relations of **M**.

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In fact:

Classical Fact (Geiger 1968, Bodnarčuk-Kalužnin-Kotov-Romov 1969)

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$$= \mathsf{SP}(\mathbb{M}).$$

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Dictionary

structure	algebra
base **M**	associated $\mathbb{M}$
pp-def. relation R	algebra $\mathbb{R} \in \mathsf{SP}(\mathbb{M})$
pp-def. equivalence relation on R	congruence of $\mathbb{R}$
pp-def. quotient R/E	quotient algebra $\mathbb{R}/E \in \mathsf{HSP}(\mathbb{M})$
pp-def. function	homomorphism
pp-interp. structure $(N, \underset{k\text{-ary}}{\underset{\uparrow}{R}})$	$\mathbb{N} \in \mathsf{HSP}(\mathbb{M})$ with $\mathbb{R} \leq \mathbb{N}^k$
$\mathbf{M}_{3SAT} \overset{pp}{\not\to} \mathbf{M}$	?

Theorem 1 (Taylor ‘77 + Hobby-McKenzie ‘88 + Bulatov-Jeavons-Krokhin ‘05 + Maróti-McKenzie ‘08 + Siggers ‘10 + Barto-Kozik ‘12)

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3. $\mathbb{M}$ has an "interesting" (*Taylor*) operation[1].

---

[1] An operation f satisfying a system Σ of one or more identities, each of the form $f(\text{variables}) = f(\text{variables})$, nontrivial in that Σ can't be modeled by $f =$ projection on $\{0,1\}$.

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2. $\neg\exists\, \mathbb{N} \in \mathsf{HSP}(\mathbb{M})$ with $N = \{0,1\}$ and $R_{3SAT} \leq \mathbb{N}^6$ (all ops of $\mathbb{N}$ are proj's).
3. $\mathbb{M}$ has an "interesting" (*Taylor*) operation[1].
4. For some $n > 1$, $\mathbb{M}$ has a *cyclic* operation $c(x_1, \ldots, x_n)$, i.e.,
$$c(x_1, x_2, \ldots, x_n) = c(x_2, \ldots, x_n, x_1) \quad \forall x_1, \ldots, x_n \in M.$$
5. $\mathbb{M}$ has a *Siggers* operation $s(x_1, \ldots, x_6)$, i.e., satisfying
$$s(x,x,y,y,z,z) = s(y,z,z,x,x,y) \quad \forall x,y,z \in M.$$

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[1] An operation f satisfying a system Σ of one or more identities, each of the form $f(\text{variables}) = f(\text{variables})$, nontrivial in that Σ can't be modeled by f = projection on $\{0,1\}$.

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(5) $\mathbb{M}$ has an operation $s(x_1, \ldots, x_6)$ satisfying

$$s(x,x,y,y,z,z) = s(y,z,z,x,x,y).$$

Proof sketch of (1) $\Longleftrightarrow$ (5) (Siggers).

($\Longleftarrow$)

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Let $\mathbf{N} = \mathbf{M}_{3SAT} = (\{0,1\},\ R_{3SAT})$.

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$s^{\mathbb{N}}$ satisfies the identity in (5), so cannot be a projection.

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Let E be the subuniverse of $\mathbb{F}^2$ generated by

$$\binom{x}{y}, \binom{x}{z}, \binom{y}{z}, \binom{y}{x}, \binom{z}{x}, \binom{z}{y}.$$

Symmetry of the generators $\implies$ E is symmetric (as a relation).

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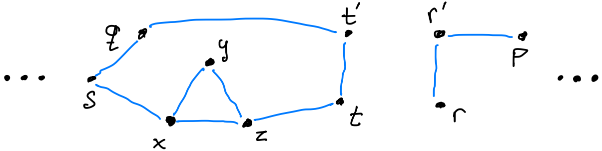
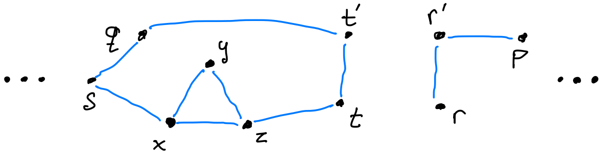
Let $\mathbf{G} = (F, E)$ (a structure, one binary relation).

Observe: $\mathbb{F} \in \mathsf{HSP}(\mathbb{M})$, $E \leq \mathbb{F}^2 \implies \mathbf{G} \overset{pp}{\hookrightarrow} \mathbf{M}$.

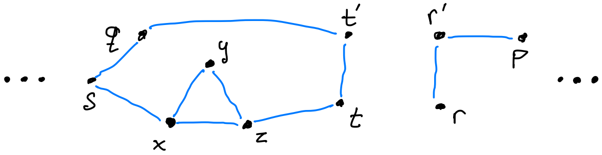


$\mathbf{G} = (F, E)$



$$\mathbf{G} = (F, E)$$

Case 1: E is irreflexive, i.e., $(p, p) \notin E$ for all $p \in F$.



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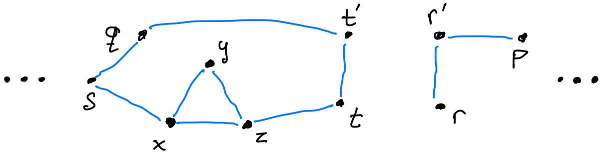
Then **G** is a (simple) graph, and non-bipartite.



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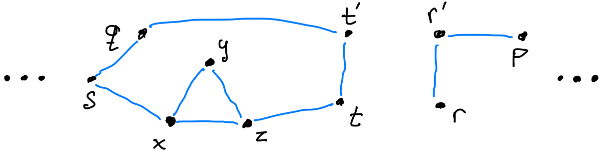
$$\mathbf{K}_3 \overset{pp}{\hookrightarrow} \mathbf{G}.$$



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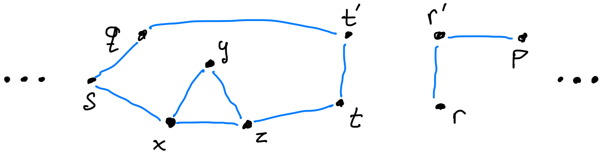

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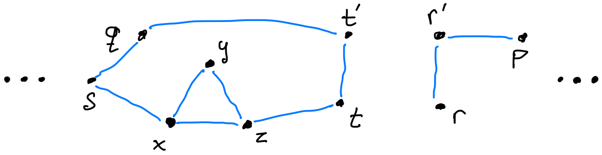

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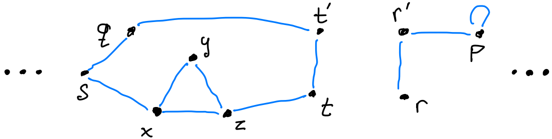
By construction, $\mathbf{G} \xhookrightarrow{pp} \mathbf{M}$.

$\therefore\ \mathbf{M}_{3SAT} \xhookrightarrow{pp} \mathbf{M}$, contrary to assumption (1).

So Case 1 is impossible: there exists a loop $(p, p) \in E$.



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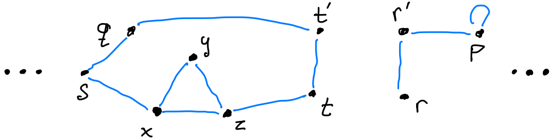

$\mathbf{G} = (F, E)$

Recall: E is generated by $\binom{x}{y}, \binom{x}{z}, \binom{y}{z}, \binom{y}{x}, \binom{z}{x}, \binom{z}{y}$.

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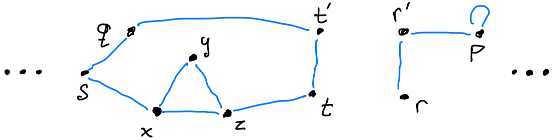

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Recall: E is generated by $\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ z \end{pmatrix}, \begin{pmatrix} y \\ z \end{pmatrix}, \begin{pmatrix} y \\ x \end{pmatrix}, \begin{pmatrix} z \\ x \end{pmatrix}, \begin{pmatrix} z \\ y \end{pmatrix}$.

$\Longrightarrow$ $\exists$ 6-ary term[1] $s(x_1, \ldots, x_6)$ such that

$$\begin{pmatrix} p \\ p \end{pmatrix} = s^{\mathbb{F}^2}\left(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ z \end{pmatrix}, \begin{pmatrix} y \\ z \end{pmatrix}, \begin{pmatrix} y \\ x \end{pmatrix}, \begin{pmatrix} z \\ x \end{pmatrix}, \begin{pmatrix} z \\ y \end{pmatrix}\right).$$

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[1] In the signature of $\mathbb{M}$, hence equal mod HSP($\mathbb{M}$) to an operation of $\mathbb{M}$.

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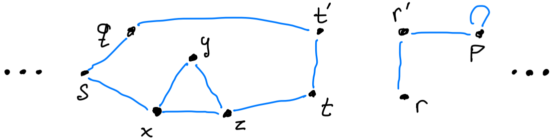

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Recall: E is generated by $\binom{x}{y}, \binom{x}{z}, \binom{y}{z}, \binom{y}{x}, \binom{z}{x}, \binom{z}{y}$.

$\implies \exists$ 6-ary term[1] $s(x_1, \dots, x_6)$ such that

$$\binom{p}{p} = s^{\mathbb{F}^2}\left(\binom{x}{y}, \binom{x}{z}, \binom{y}{z}, \binom{y}{x}, \binom{z}{x}, \binom{z}{y}\right).$$

A standard argument gives $\mathbb{M} \models s(x, x, y, y, z, z) = s(y, z, z, x, x, y)$. ☐

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Summary of Lecture 1

$\mathrm{CSP}_p(\mathbf{M})$: decision problem about satisfiability of $\wedge$at-fmlas/$\mathbf{M}$.

CSP Dichotomy Theorem of Bulatov and Zhuk (2017, 2020):

$$\mathbf{M}_{3SAT} \overset{pp}{\not\to} \mathbf{M} \implies \mathrm{CSP}_p(\mathbf{M}) \text{ is in P.}$$

Algebraic perspective

- $\mathbf{M} \mapsto$ idempotent polymorphism algebra $\mathbb{M}$.
- Connections between $\mathrm{HSP}(\mathbb{M})$ and pp-definable relations over $\mathbf{M}$.

Positive characterization of $\mathbf{M}_{3SAT} \overset{pp}{\not\to} \mathbf{M}$ (Theorem 1):

"$\mathbb{M}$ has a Taylor operation"